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Basis free relations for the conjugate stresses of the strains based on the right stretch tensor

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Abstract

In this paper, general relations between two different stress tensors $\mathbf{T}^{\mathrm{f}}$ and $\mathbf{T}^{\mathrm{g}}$, respectively conjugate to strain measure tensors $f(\mathbf{U})$ and $g(\mathbf{U})$ are found. The strain class $f(\mathbf{U})$ is based on the right stretch tensor $\mathbf{U}$ which includes the Seth–Hill strain tensors. The method is based on the definition of energy conjugacy and Hill's principal axis method. The relations are derived for the cases of distinct as well as coalescent principal stretches. As a special case, conjugate stresses of the Seth–Hill strain measures are then more investigated in their general form. The relations are first obtained in the principal axes of the tensor $\mathbf{U}$. Then they are used to obtain basis free tensorial equations between different conjugate stresses. These basis free equations between two conjugate stresses are obtained through the comparison of the relations between their components in the principal axes, with a possible tensor expansion relation between the stresses with unknown coefficients, the unknown coefficients to be obtained. In this regard, some relations are also obtained for $\mathbf{T}^{(0)}$ which is the stress conjugate to the logarithmic strain tensor $\ln\mathbf{U}$.

Keywords: Energy conjugacy; Basis free; Seth–Hill strains

1. Introduction

The concept of energy conjugacy first presented by Hill (1968) states that a stress measure $\mathbf{T}$ is said to be conjugate to a strain measure $\mathbf{E}$ if $\mathbf{T}:\dot{\mathbf{E}}$ represents power or rate of change of internal energy per unit reference volume, $\dot{w}$. That is

$$\dot{w} = III\boldsymbol{\sigma}:\mathbf{D} = \mathbf{T}:\dot{\mathbf{E}} \tag{1.1}$$

where $\boldsymbol{\sigma}$ and $\mathbf{D}$ are Cauchy stress and strain rate tensors, respectively, $III = \det(\mathbf{U})$ is the third invariant of the right stretch tensor $\mathbf{U}$, and $(\dot{\ })$ is material time derivative operator.

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According to the spectral decomposition theorem

$$U = \sum_i \lambda_i N_i \otimes N_i \tag{1.2}$$

where λ_i and N_i are the principal stretches and corresponding orthonormal eigenvectors of the second order tensor $\mathbf{U}$, respectively.

The Seth–Hill class of strain measure tensors $\mathbf{E}^{(m)}$ (Seth, 1964; Hill, 1968) indexed by superscript m is defined as

$$\mathbf{E}^{(m)} = \frac{1}{m}\sum_i (\lambda_i^m - 1) N_i \otimes N_i = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}); \quad m \neq 0 \tag{1.3a}$$

$$\mathbf{E}^{(0)} = \sum_i \ln(\lambda_i) N_i \otimes N_i = \ln\mathbf{U} \tag{1.3b}$$

where $\mathbf{I}$ is the identity tensor. Guo and Man (1992) derived explicit tensorial formulations for conjugate stresses $\mathbf{T}^{(m)}$ for $|m| \geqslant 3$, whilst earlier, the stress measure conjugate to logarithmic strain tensor $\ln\mathbf{U}$, had been derived by Hoger (1987).

A more general class of strain measures based on the right stretch tensor $\mathbf{U}$, was cited by Hill (1968) as

$$f(\mathbf{U}) = \sum_i f(\lambda_i) N_i \otimes N_i \tag{1.4a}$$

where $f(\,)$ is a smooth and strictly increasing scalar function that meets the conditions

$$f(1) = 0, \quad f'(1) = 1, \quad \text{and} \quad f'(\lambda) > 0 \tag{1.4b}$$

Following the Hill's principal axis method and energy conjugacy notion, a method was proposed (Farahani and Naghdabadi, 2000) to find the relation between the components of two Seth–Hill conjugate stress tensors in the principal axes N_i of the right stretch tensor $\mathbf{U}$. In that work, assuming the index m in Eq. (1.3a) was a positive or negative non-zero integer, the material time derivative of Seth–Hill strains for both these cases were expanded as

$$\dot{\mathbf{E}}^{(m)} = \frac{1}{|m|}\sum_{r=1}^{|m|} \mathbf{U}^{s(m-r)} \dot{\overline{\mathbf{U}^s}} \mathbf{U}^{s(r-1)} \tag{1.5}$$

where $s = \text{sign}(m) = |m|/m$.

Using the Hill's principal axis method, the stress tensor $\mathbf{T}^{(m)}$ conjugate to the strain measure $\mathbf{E}^{(m)}$ can be introduced in the principal axes as

$$\mathbf{T}^{(m)} = \sum_{i,j} \mathbf{T}_{ij}^{(m)} N_i \otimes N_j \tag{1.6}$$

Writing (1.1) for two different conjugate pairs, we have

$$\mathbf{T}^{(m)} : \dot{\mathbf{E}}^{(m)} = \mathbf{T}^{(n)} : \dot{\mathbf{E}}^{(n)} \tag{1.7}$$

Noting that $\mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$, and $\dot{\overline{\mathbf{U}^{-1}}} = -\mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1}$, and substituting (1.5) and (1.6) in (1.7) for non-coalescent stretches resulted in

$$\begin{cases} \mathbf{T}_{ii}^{(n)} = \mathbf{T}_{ii}^{(m)} \lambda_i^{m-n} \\ \mathbf{T}_{ij}^{(n)} = \dfrac{n}{m}\mathbf{T}_{ij}^{(n)} \dfrac{\lambda_i^m - \lambda_j^m}{\lambda_i^n - \lambda_j^n}; \quad i \neq j \end{cases} \tag{1.8a,b}$$

where m and n may be positive and negative non-zero integers. The case of coalescent principal stretches was considered in that work as well.

The main aim of this paper is to

1. Extend our previous work to obtain general equations similar to (1.8), for the stresses conjugate to two wide range of strain measures
   (a) Strain measures presented in (1.3), for every real index (m), including zero which corresponds to the logarithmic strain $\ln(\mathbf{U})$.
   (b) Strain measures with the more general form $f(\mathbf{U})$ presented in (1.4).
2. Obtain basis free tensor equations between two different stress measures, using the relation between their components in the principal axes.

Hence, some equations will be derived for the principal components of conjugate stresses for not only the Seth–Hill strain measures including logarithmic strain, but also for all the strain measures in the form of Eq. (1.4). Using these relations, several basis free equalities are obtained, some of which have been obtained already through a different approach.

In this work, the index notation is not used unless stated otherwise. Second order tensors are in bold capitals and fourth order tensors are in italic bold capitals.

In what follows, two different approaches are adopted to obtain the formulas for the principal components of the stresses. The method presented in Section 3 is based on the arbitrariness of Lagrangian spin tensor components, which is less general since there are instances that some of the components of the spin tensor are not well defined. The second one presented in Section 4 is based on the tensor algebra and is quite general and covers all cases.

2. Basic relations

Consider a deforming body, with $\mathbf{F}$ denoting the deformation gradient at a point inside it with $\det(\mathbf{F}) > 0$. The polar decomposition theorem states that $\mathbf{F}$ may uniquely be decomposed as

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{2.1}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively, and are both positive definite symmetric tensors, and $\mathbf{R}$ is a proper orthogonal rotation tensor. Here, N_i and n_i are the principal axes or eigenvectors of $\mathbf{U}$ and $\mathbf{V}$, respectively, and

$$n_i = \mathbf{R}N_i \tag{2.2}$$

Therefore, Eq. (2.1) states that a material finite deformation can be viewed as a pure stretch along a specific orthogonal Lagrangian triad N_i, followed by a rigid rotation of this orthogonal triad into another specific orthogonal Eulerian triad n_i, or conversely, a rigid rotation followed by a pure stretch.

The eigenvalues of $\mathbf{U}$ and $\mathbf{V}$ called principal stretches, are denoted by λ_1, λ_2 and λ_3. The principal invariants of $\mathbf{U}$ and $\mathbf{V}$ are

$$\begin{aligned} I &= \lambda_1 + \lambda_2 + \lambda_3 \\ II &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ III &= \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{2.3}$$

The Cayley–Hamilton theorem declares that every tensor satisfies its own characteristic equation. That is, for the second order tensor $\mathbf{U}$

$$\mathbf{U}^3 - I\mathbf{U}^2 + II\mathbf{U} - III\mathbf{I} = \mathbf{0} \tag{2.4}$$

During the deformation, the continually changing Lagrangian and Eulerian triads and the rigid rotation of the material will have the spins $\mathbf{\Omega}^{\mathrm{L}}$, $\mathbf{\Omega}^{\mathrm{E}}$, and $\mathbf{\Omega}^{\mathrm{R}}$. Some basic relations between these spin tensors are (Mehrabadi and Nemat-Nasse, 1987)

$$\dot{N}_i = \mathbf{\Omega}^{\mathrm{L}} N_i \tag{2.5}$$

$$\dot{n}_i = \mathbf{\Omega}^{\mathrm{E}} n_i \tag{2.6}$$

$$\dot{\mathbf{R}} = \mathbf{\Omega}^{\mathrm{R}} \mathbf{R} \tag{2.7}$$

In these relations, $\mathbf{\Omega}$'s are anti-symmetric spin tensors which are related through

$$\mathbf{\Omega}^{\mathrm{E}} = \mathbf{\Omega}^{\mathrm{R}} + \mathbf{R}\mathbf{\Omega}^{\mathrm{L}}\mathbf{R}^{\mathrm{T}} \tag{2.8}$$

According to (2.5), the material time derivative of (1.2) and (1.4) can be written as

$$\dot{\mathbf{U}} = \sum_i \dot{\lambda}_i N_i \otimes N_i + \mathbf{\Omega}^{\mathrm{L}}\mathbf{U} - \mathbf{U}\mathbf{\Omega}^{\mathrm{L}} \tag{2.9}$$

$$\dot{f}(\mathbf{U}) = \sum_i \dot{\lambda}_i f'(\lambda_i) N_i \otimes N_i + \mathbf{\Omega}^{\mathrm{L}} f(\mathbf{U}) - f(\mathbf{U})\mathbf{\Omega}^{\mathrm{L}} \tag{2.10}$$

where $()'$ means derivative with respect to λ.

Some of the well-known relations of the Seth–Hill strain measures with their conjugate stresses are as follows (Hill, 1978; Guo and Dubey, 1984)

(i) Green's strain and second Piola–Kirchhoff stress tensors

$$\mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}); \quad \mathbf{T}^{(2)} = III\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} \tag{2.11}$$

(ii) Nominal strain and Jaumann stress tensors, alternatively called Biot strain and stress tensors (Ogden, 1984)

$$\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}; \quad \mathbf{T}^{(1)} = \tfrac{1}{2}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}) \tag{2.12}$$

(iii) The conjugate pairs $\mathbf{T}^{(-1)}$ and $\mathbf{E}^{(-1)}$ (Guo and Man, 1992)

$$\mathbf{E}^{(-1)} = \mathbf{I} - \mathbf{U}^{-1}; \quad \mathbf{T}^{(-1)} = \tfrac{1}{2}(\mathbf{T}^{(-2)}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{T}^{(-2)}) \tag{2.13}$$

(iv) Almansi strain and the weighted convected stress tensors

$$\mathbf{E}^{(-2)} = \frac{1}{2}(\mathbf{I} - \mathbf{U}^{-2}); \quad \mathbf{T}^{(-2)} = III\mathbf{F}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{F} \tag{2.14}$$

(v) Logarithmic strain $\mathbf{E}^{(0)} = \ln(\mathbf{U})$ and its conjugate $\mathbf{T}^{(0)}$ (Hoger, 1987).

(vi) Seth–Hill conjugate stresses with opposite index signs (Farahani and Naghdabadi, 2000)

$$\mathbf{T}^{(-n)} = \mathbf{U}^n \mathbf{T}^{(n)} \mathbf{U}^n \tag{2.15}$$

3. Relations between conjugate stresses of the strain measures *f*(U)

In this section, the relations between the stress tensors energetically conjugate to $f(\mathbf{U})$ defined in (1.4), are obtained. Then, the results will be used to find similar equations for the stresses conjugate to the Seth–Hill strain tensors and the logarithmic strain $\ln\mathbf{U}$.

Considering a deforming body in the current configuration, let the stress measure conjugate to $f(\mathbf{U})$ be $\mathbf{T}^{\mathrm{f}}$. Writing the stress tensor $\mathbf{T}^{\mathrm{f}}$ in the orthonormal basis N_i, we have

$$\mathbf{T}^{\mathrm{f}} = \sum_{ij} \mathbf{T}^{\mathrm{f}}_{ij} N_i \otimes N_j \tag{3.1}$$

where $\mathbf{T}^{\mathrm{f}}_{ij}$ are the principal components of the tensor $\mathbf{T}^{\mathrm{f}}$. According to (1.1), the power or the rate of change of internal energy per unit reference volume of a deforming body can be written in terms of two different strain tensors $f(\mathbf{U})$ and $g(\mathbf{U})$, and their conjugate stresses $\mathbf{T}^{\mathrm{f}}$ and $\mathbf{T}^{\mathrm{g}}$

$$\mathbf{T}^{\mathrm{f}} : \dot{f}(\mathbf{U}) = \mathbf{T}^{\mathrm{g}} : \dot{g}(\mathbf{U}) \tag{3.2}$$

where $g()$ is also a smooth function which satisfies the same conditions as $f()$ does. Substitution of (2.10) and (3.1) into (3.2) yields

$$\begin{aligned} &\left(\sum_{ij} \mathbf{T}^{\mathrm{f}}_{ij} N_i \otimes N_j\right) : \left(\sum_{i} \left(\dot{\lambda}_i f'(\lambda_i) N_i \otimes N_i\right) + \mathbf{\Omega}^{\mathrm{L}} f(\mathbf{U}) - f(\mathbf{U})\mathbf{\Omega}^{\mathrm{L}}\right) \\ &\quad = \left(\sum_{ij} \mathbf{T}^{\mathrm{g}}_{ij} N_i \otimes N_j\right) : \left(\sum_{i} \left(\dot{\lambda}_i g'(\lambda_i) N_i \otimes N_i\right) + \mathbf{\Omega}^{\mathrm{L}} g(\mathbf{U}) - g(\mathbf{U})\mathbf{\Omega}^{\mathrm{L}}\right) \end{aligned} \tag{3.3}$$

Consider the spin tensor $\mathbf{\Omega}^{\mathrm{L}}$ presented in the principal axes N_i such that

$$\mathbf{\Omega}^{\mathrm{L}} = \sum_{ij} \mathbf{\Omega}^{\mathrm{L}}_{ij} N_i \otimes N_j \tag{3.4}$$

All the tensors involved in (3.3) are now defined in the principal axes. Therefore, substituting (1.4) and (3.4) into (3.3) and rearranging the equation result in

$$\sum_{i} \left\{ \dot{\lambda}_i \left(f'(\lambda_i)\mathbf{T}^{\mathrm{f}}_{ii} - g'(\lambda_i)\mathbf{T}^{\mathrm{g}}_{ii} \right) \right\} + \sum_{ij} \left\{ \mathbf{\Omega}^{\mathrm{L}}_{ij} \left\{ \mathbf{T}^{\mathrm{f}}_{ij}\left(f(\lambda_j) - f(\lambda_i)\right) - \mathbf{T}^{\mathrm{g}}_{ij}\left(g(\lambda_j) - g(\lambda_i)\right) \right\} \right\} = 0 \tag{3.5}$$

Since in Eq. (3.5), the stretch rates $\dot{\lambda}_i$ and spin tensor components $\mathbf{\Omega}^{\mathrm{L}}_{ij}$ are arbitrary, their coefficients must be equal to zero.

3.1. The case of non-coalescent principal stretches

By setting the coefficients in (3.5) equal to zero, we arrive at the following general equalities

$$\begin{cases} \mathbf{T}^{\mathrm{f}}_{ii} = \dfrac{g'(\lambda_i)}{f'(\lambda_i)} \mathbf{T}^{\mathrm{g}}_{ii}; & i = 1, 2, 3 \\ \mathbf{T}^{\mathrm{f}}_{ij} = \dfrac{g(\lambda_i) - g(\lambda_j)}{f(\lambda_i) - f(\lambda_j)} \mathbf{T}^{\mathrm{g}}_{ij}; & i \neq j \end{cases} \tag{3.6a,b}$$

Eqs. (3.6) give the relation between the principal components of two different conjugate stress tensors. These equations will later be used to obtain bases free equations for several stress tensors. Eq. (3.6b) is obviously only for the case of non-coalescent eigen stretches.

It is noted that this approach to obtain Eqs. (3.6) is acceptable only when the Lagrangian spin tensor $\mathbf{\Omega}^{\mathrm{L}}$ is well defined. However, there are situations where some of the components of $\mathbf{\Omega}^{\mathrm{L}}$ tend to infinity in an instant at the boundary of two adjacent intervals where $\mathbf{U}$ has a different number of distinct eigenvalues over the two intervals (Guo et al., 1992). In this case where $\mathbf{\Omega}^{\mathrm{L}}$ is not well defined, a different approach can be used which is explained in Section 4 of the paper.

3.2. The case of two coalescent principal stretches

From Eq. (3.6) it is observed that coincidence of principal stretches just affects the relation between the off-diagonal members of the stress components. Therefore, for two coalescent principal stretches, Eq. (3.6b) can be modified as

$$\mathbf{T}^{\mathrm{f}}_{ij} = \lim_{\lambda_i \to \lambda_j} \left(\frac{g(\lambda_i) - g(\lambda_j)}{f(\lambda_i) - f(\lambda_j)} \right) \mathbf{T}^{\mathrm{g}}_{ij} = \frac{g'(\lambda_i)}{f'(\lambda_i)} \mathbf{T}^{\mathrm{g}}_{ij}; \quad i \neq j \tag{3.7}$$

Hence, for two coalescent principal stretches we arrive at

$$\begin{cases} \mathbf{T}^{\mathrm{f}}_{ij} = \dfrac{g'(\lambda_i)}{f'(\lambda_i)} \mathbf{T}^{\mathrm{g}}_{ij}; & (i = j) \text{ or } (\lambda_i = \lambda_j) \\ \mathbf{T}^{\mathrm{f}}_{ij} = \dfrac{g(\lambda_i) - g(\lambda_j)}{f(\lambda_i) - f(\lambda_j)} \mathbf{T}^{\mathrm{g}}_{ij}; & (i \neq j) \ \& \ (\lambda_i \neq \lambda_j) \end{cases} \tag{3.8a,b}$$

3.3. The case of three coalescent principal stretches

Similarly, using the same method of Section (3.2), it is concluded that for three coalescent principal stretches, we have

$$\left\{ \mathbf{T}^{\mathrm{f}}_{ij} = \frac{g'(\lambda)}{f'(\lambda)} \mathbf{T}^{\mathrm{g}}_{ij}; \quad (\lambda_1 = \lambda_2 = \lambda_3 = \lambda) \right. \tag{3.9}$$

That is, in the case of three coalescent principal stretches, all the stress tensors conjugate to the strains in the form of $f(\mathbf{U})$ defined in (1.4) are coaxial. Eqs. (3.6–3.9) will later be used for the special case of the Seth–Hill satin tensors.

4. General proof for the relation between conjugate stresses

The general proof for Eqs. (3.6) is briefly explained in this section. Index notation is not used in this section unless stated otherwise. Fourth order tensors are in bold italic capitals. The space of all three dimensional real vectors is denoted by Vect, and the space of all second order tensors which are linear transformation from Vect into Vect is called **Lin**. Furthermore, the fourth order tensors constitutes the space of all linear mappings of **Lin** into itself, called **Lin**. The double contraction of a fourth tensor $\boldsymbol{D} \in \mathbf{Lin}$ and a second order tensor $\mathbf{U} \in \mathbf{Lin}$ in index notation is defined as

$$\boldsymbol{D} : \mathbf{U} = \boldsymbol{D}_{ijkl} \mathbf{U}_{jk} e_i \otimes e_l \tag{4.1}$$

where e_i's are the basis vectors. According to Truesdel and Noll (1965), Gurkin (1981), and Silhavy (1997), derivatives can be identified as linear transformations. The derivatives of a scalar valued tensor function $\alpha(\mathbf{U}) : \mathbf{Lin} \to \mathbf{IR}$, and a tensor function $f(\mathbf{U}) : \mathbf{Lin} \to \mathbf{Lin}$ in index notation are defined respectively as

$$\frac{\partial \alpha(\mathbf{U})}{\partial \mathbf{U}} = \partial_{\mathbf{U}} \alpha(\mathbf{U}) = \frac{\partial \alpha(\mathbf{U})}{\partial \mathbf{U}_{ij}} e_i \otimes e_j \tag{4.2}$$

$$\frac{\partial f(\mathbf{U})}{\partial \mathbf{U}} = \partial_{\mathbf{U}} f(\mathbf{U}) = \frac{\partial f_{ij}(\mathbf{U})}{\partial \mathbf{U}_{kl}} e_i \otimes e_k \otimes e_l \otimes e_j \tag{4.3}$$

which show that (4.2) and (4.3) are second and fourth order tensors, respectively. Using the chain rule for the differentiation in tensor derivatives, we have

$$\frac{\mathrm{d}f(\mathbf{U})}{\mathrm{d}t}=\frac{\partial f(\mathbf{U})}{\partial \mathbf{U}}:\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \tag{4.4}$$

Using the spectral decomposition, $f(\mathbf{U})$ can be expanded as

$$f(\mathbf{U})=\sum_i f(\lambda_i)N_i\otimes N_i \tag{4.5}$$

Assuming that the scalar function f is continuously differentiable, according to Prepositions 1.2.6 or 8.1.9 thoroughly explained by Silhavy (1997), we can write

$$\partial_{\mathbf{U}}f(\mathbf{U}):\mathbf{T}^{\mathrm{f}}=\sum_{i,j}H_{ij}\mathbf{T}^{\mathrm{f}}_{ij}N_i\otimes N_j \tag{4.6}$$

where $\mathbf{T}^{\mathrm{f}}_{ij}$ are the components of $\mathbf{T}^{\mathrm{f}}$ in the principal axes of $\mathbf{U}$ as defined in (3.1). The components H_{ij} are independent of $\mathbf{T}^{\mathrm{f}}$ obtained as

$$H_{ij}=\begin{cases}\dfrac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}; & \lambda_i\neq\lambda_j\\ f'(\lambda_i); & \lambda_i=\lambda_j\end{cases} \tag{4.7}$$

Recalling Eqs. (3.2) and (4.4), we have

$$\dot f(\mathbf{U}):\mathbf{T}^{\mathrm{f}}=\dot g(\mathbf{U}):\mathbf{T}^{\mathrm{g}} \tag{4.8}$$

$$\left(\frac{\partial f(\mathbf{U})}{\partial \mathbf{U}}:\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\right):\mathbf{T}^{\mathrm{f}}=\left(\frac{\partial g(\mathbf{U})}{\partial \mathbf{U}}:\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\right):\mathbf{T}^{\mathrm{g}} \tag{4.9}$$

Because of symmetry of the tensors, it is easy to show that $\mathrm{d}\mathbf{U}/\mathrm{d}t$ is commutative in (4.8) as

$$\left(\frac{\partial f(\mathbf{U})}{\partial \mathbf{U}}:\mathbf{T}^{\mathrm{f}}\right):\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}=\left(\frac{\partial g(\mathbf{U})}{\partial \mathbf{U}}:\mathbf{T}^{\mathrm{g}}\right):\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \tag{4.10}$$

Hence, since (4.10) holds for every tensor $\mathbf{U}$, it is concluded that

$$\frac{\partial f(\mathbf{U})}{\partial \mathbf{U}}:\mathbf{T}^{\mathrm{f}}=\frac{\partial g(\mathbf{U})}{\partial \mathbf{U}}:\mathbf{T}^{\mathrm{g}} \tag{4.11}$$

Therefore, substitution of (4.6) and (4.7) into (4.11), it is concluded that

$$\begin{cases}\mathbf{T}^{\mathrm{f}}_{ij}=\dfrac{g'(\lambda_i)}{f'(\lambda_i)}\mathbf{T}^{\mathrm{g}}_{ij}; & (i=j)\ \text{or}\ (\lambda_i=\lambda_j)\\ \mathbf{T}^{\mathrm{f}}_{ij}=\dfrac{g(\lambda_i)-g(\lambda_j)}{f(\lambda_i)-f(\lambda_j)}\mathbf{T}^{\mathrm{g}}_{ij}; & (i\neq j)\ \&\ (\lambda_i\neq\lambda_j)\end{cases} \tag{4.12}$$

which is the same as what was obtained earlier in (3.6) and (3.8).

5. Conjugate stresses of general Seth–Hill strain tensors $\mathbf{E}^{(\alpha)}$

5.1. The case of non-zero indices

Recalling Eqs. (1.3), we consider the strain measure $\mathbf{E}^{(\alpha)}$ and its conjugate stress $\mathbf{T}^{(\alpha)}$, where the index number α is not necessarily an integer and can be any non-zero real number. Using Eqs. (1.4), for two arbitrary non-zero real indices α and β we have

$$f(\lambda) = \frac{1}{\alpha}(\lambda^{\alpha} - 1); \quad g(\lambda) = \frac{1}{\beta}(\lambda^{\beta} - 1) \tag{5.1}$$

Making use of (3.6) or (4.12), it is concluded that

$$\begin{cases} \mathbf{T}_{ii}^{(\alpha)} = \lambda_i^{\beta-\alpha}\mathbf{T}_{ii}^{(\beta)}; & i = 1,2,3 \\ \mathbf{T}_{ij}^{(\alpha)} = \dfrac{\alpha}{\beta}\dfrac{\lambda_i^{\beta} - \lambda_j^{\beta}}{\lambda_i^{\alpha} - \lambda_j^{\alpha}}\mathbf{T}_{ij}^{(\beta)}; & i \neq j \end{cases} \tag{5.2a,b}$$

Eqs. (5.2) is exactly the same as (1.8) which had been developed earlier only for integer indices.

*5.2. The case of zero index or the logarithmic strain ln(**U**)*

To find similar relations between $\mathbf{T}^{(0)}$, the conjugate stress of the logarithmic strain $\ln(\mathbf{U})$, and other Seth–Hill strain conjugates, we have

$$f(\lambda) = \ln(\lambda); \quad g(\lambda) = \frac{1}{\alpha}(\lambda^{\alpha} - 1) \tag{5.3}$$

Again, by making use of (3.6), we arrive at

$$\begin{cases} \mathbf{T}_{ii}^{(0)} = \lambda_i^{\alpha}\mathbf{T}_{ii}^{(\alpha)}; & i = 1,2,3 \\ \mathbf{T}_{ij}^{(0)} = \dfrac{1}{\alpha}\dfrac{\lambda_i^{\alpha} - \lambda_j^{\alpha}}{\ln\left(\dfrac{\lambda_i}{\lambda_j}\right)}\mathbf{T}_{ij}^{(\alpha)}; & i \neq j \end{cases} \tag{5.4a, b}$$

Multiplying (5.4) by $N_i \otimes N_j$ and summing over i and j, we arrive at the following basis free tensor equation

$$\mathbf{T}^{(0)}\ln\mathbf{U} - \ln\mathbf{U}\mathbf{T}^{(0)} = \mathbf{T}^{(\alpha)}\mathbf{E}^{(\alpha)} - \mathbf{E}^{(\alpha)}\mathbf{T}^{(\alpha)} \tag{5.5}$$

It is noted that Hill (1978) obtained Eq. (5.5) for every conjugate stress and strain pairs.

5.3. The case of equal principal stretches

For the case of three coalescent principal stretches where $\mathbf{U} = \lambda\mathbf{I}$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, from (3.9) it is concluded that

$$\mathbf{T}_{ij}^{(0)} = \lambda^{\alpha}\mathbf{T}_{ij}^{(\alpha)} \tag{5.6}$$

or in the basis free form

$$\mathbf{T}^{(\alpha)} = \lambda\mathbf{T}^{(\alpha+1)} \tag{5.7}$$

6. Application to basis free tensor equations

The application of the above formulae in deriving basis free tensor equations is presented here which may be used to obtain basis free relations between any two different stress tensors conjugate to the class of strains of the form (1.4).

6.1. The stress $\mathbf{T}^{(0)}$ conjugate to the logarithmic strain ln $\mathbf{U}$, and $\mathbf{T}^{(1)}$

The expression for the stress conjugate to logarithmic strain has been already obtained in previous works (Hoger, 1987). Here, we obtain a basis free relation for it through another approach. A basis free relation will be obtained between $\mathbf{T}^{(0)}$ the stress conjugate to the logarithmic strain ln$\mathbf{U}$, and the Biot stress tensor $\mathbf{T}^{(1)}$. From (5.4) we can write

$$\begin{cases} \mathbf{T}^{(0)}_{ii} = \lambda_i \mathbf{T}^{(1)}_{ii}; & i = 1,2,3 \\ \mathbf{T}^{(0)}_{ij} = \dfrac{\lambda_i - \lambda_j}{\ln\left(\dfrac{\lambda_i}{\lambda_j}\right)} \mathbf{T}^{(1)}_{ij}; & i \neq j \end{cases} \tag{6.1a,b}$$

We can expand $\mathbf{T}^{(0)}$ in a symmetric basis free form in terms of $\mathbf{T}^{(1)}$

$$\mathbf{T}^{(0)} = A_1\mathbf{T}^{(1)} + A_2\left(\mathbf{U}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}\right) + A_3\mathbf{U}\mathbf{T}^{(1)}\mathbf{U} + A_4\left(\mathbf{U}^2\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}^2\right) + A_5\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}^2 + A_6\left(\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(1)}\mathbf{U}^2\right) \tag{6.2}$$

It is noted that (6.2) is not the only possible expansion of $\mathbf{T}^{(0)}$ in terms of $\mathbf{T}^{(1)}$. Since (6.2) is a basis free equation, the coefficients A_i can be obtained in any coordinates. We may obtain the coefficients A_i especially in the principal axes, where $\mathbf{U}$ is diagonal and is expressed only in terms of stretches, by comparing (6.1) and (6.2) on the principal axes. This requires the solution of a 6×6 system of parametric equations, and rearranging the obtained coefficients in terms of the three invariants. Hence, the coefficients of (6.2) may be presented as

$$A_1 = \frac{2III}{L}\sum_i \frac{\lambda_i}{\ln\dfrac{\lambda_j}{\lambda_k}} + \frac{III}{L^2}(3I\cdot III + II\cdot I^2 - 4II^2) \tag{6.3a}$$

$$A_2 = \frac{-1}{L}\sum_i \frac{III + \lambda_i II}{\ln\dfrac{\lambda_j}{\lambda_k}} + \frac{III}{L^2}(9III - 7II\cdot I + 2I^3) \tag{6.3b}$$

$$A_3 = \frac{2}{L}\sum_i \frac{II + \lambda_i^2}{\ln\dfrac{\lambda_j}{\lambda_k}} + \frac{1}{L^2}(II\cdot I - III)(I^2 - 3II) \tag{6.3c}$$

$$A_4 = \frac{1}{L}\sum_i \frac{II - \lambda_j\lambda_k}{\ln\dfrac{\lambda_j}{\lambda_k}} + \frac{2III}{L^2}(I^2 - 3II) \tag{6.3d}$$

$$A_5 = \frac{2}{L}\sum_i \frac{1}{\ln\dfrac{\lambda_j}{\lambda_k}} + \frac{1}{L^2}(II\cdot I - 9III) \tag{6.3e}$$

$$A_6 = \frac{-1}{L}\sum_i \frac{I + \lambda_i}{\ln\dfrac{\lambda_j}{\lambda_k}} + \frac{1}{L^2}(3III\cdot I + 2II^2 - I^2\cdot II) \tag{6.3f}$$

where $i = 1$, 2, 3, and

$$L = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \tag{6.3g}$$

In the Eqs. (6.3), j and k are set by permutation.

Writing $\mathbf{T}^{(1)}$ in terms of $\mathbf{T}^{(0)}$ in a basis free form gives rise to more complicated coefficients. For this purpose we similarly expand $\mathbf{T}^{(1)}$ as

$$\mathbf{T}^{(1)} = B_1\mathbf{T}^{(0)} + B_2\left(\mathbf{U}\mathbf{T}^{(0)} + \mathbf{T}^{(0)}\mathbf{U}\right) + B_3\mathbf{U}\mathbf{T}^{(0)}\mathbf{U} + B_4\left(\mathbf{U}^2\mathbf{T}^{(0)} + \mathbf{T}^{(0)}\mathbf{U}^2\right) + B_5\mathbf{U}^2\mathbf{T}^{(0)}\mathbf{U}^2 + B_6\left(\mathbf{U}^2\mathbf{T}^{(0)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(0)}\mathbf{U}^2\right) \tag{6.4}$$

From (6.1) we have

$$\begin{cases} \mathbf{T}^{(1)}_{ii} = \dfrac{\mathbf{T}^{(0)}_{ii}}{\lambda_i}; & i = 1,2,3 \\ \mathbf{T}^{(1)}_{ij} = \dfrac{\ln\left(\dfrac{\lambda_i}{\lambda_j}\right)}{\lambda_i - \lambda_j}\mathbf{T}^{(0)}_{ij}; & i \neq j \end{cases} \tag{6.5a,b}$$

Again comparing (6.4) and (6.5) on the principal axes, solving the resulted 6×6 system of equation for B_i, and rearranging them in terms of invariants of $\mathbf{U}$, we arrive at

$$B_1 = \frac{2III}{L}\sum_i \frac{\lambda_i \ln\dfrac{\lambda_j}{\lambda_k}}{(\lambda_j - \lambda_k)^2} - \frac{1}{III \cdot L^2}\left(3I^3 \cdot II \cdot III - I^2 \cdot II^3 + I^2 \cdot III^2 - 14I \cdot II^2 \cdot III + 4II^4 + 15II \cdot III^2\right) \tag{6.6a}$$

$$B_2 = \frac{-1}{L}\sum_i \frac{(\lambda_i II + III)\ln\dfrac{\lambda_j}{\lambda_k}}{(\lambda_j - \lambda_k)^2} + \frac{1}{III \cdot L^2}\left(2I^4 \cdot III - I^3 \cdot II^2 - 9I^2 \cdot II \cdot III + 4I(II^3 + 3III^2) - 4II^2 \cdot III\right) \tag{6.6b}$$

$$B_3 = \frac{2}{L}\sum_i \frac{(\lambda_i^2 + II)\ln\dfrac{\lambda_j}{\lambda_k}}{(\lambda_j - \lambda_k)^2} + \frac{1}{III \cdot L^2}(I \cdot II - III)(I^3 + 9III - 4I \cdot II) \tag{6.6c}$$

$$B_4 = \frac{1}{L}\sum_i \frac{(II - \lambda_j\lambda_k)\ln\dfrac{\lambda_j}{\lambda_k}}{(\lambda_j - \lambda_k)^2} - \frac{1}{III \cdot L^2}\left(2I^3 \cdot III - I^2 \cdot II^2 - 10I \cdot II \cdot III + 4II^3 + 9III^2\right) \tag{6.6d}$$

$$B_5 = \frac{2}{L}\sum_i \frac{\ln\dfrac{\lambda_j}{\lambda_k}}{(\lambda_j - \lambda_k)^2} - \frac{1}{III \cdot L^2}(I^2 \cdot II - 4II^2 + 3I \cdot III) \tag{6.6e}$$

$$B_6 = \frac{-1}{L}\sum_i \frac{(I + \lambda_i)\ln\dfrac{\lambda_j}{\lambda_k}}{(\lambda_j - \lambda_k)^2} - \frac{1}{III \cdot L^2}\left(I^3 \cdot II + I^2 \cdot III - 4I \cdot II^2 + 6II \cdot III\right) \tag{6.6f}$$

where L is defined in (6.3g).

For the case of coalescent principal stretches, the coefficients take simpler forms and we skip this issue for the sake of brevity.

6.2. The Biot stress $\boldsymbol{T}^{(1)}$ conjugate to the nominal strain $(\boldsymbol{U}-\boldsymbol{I})$, and $\boldsymbol{T}^{(2)}$

The expression for the Biot stress $\mathbf{T}^{(1)}$ in terms of second Piola–Kirchhoff stress tensor $\mathbf{T}^{(2)}$ is quite well known. However, it will be obtained here again. From (5.2) we have

$$\begin{cases} \mathbf{T}^{(1)}_{ii} = \lambda_i \mathbf{T}^{(2)}_{ii}; & i = 1,2,3 \\ \mathbf{T}^{(1)}_{ij} = \dfrac{1}{2}\dfrac{\lambda_i^2-\lambda_j^2}{\lambda_i-\lambda_j}\mathbf{T}^{(2)}_{ij}; & i \neq j \end{cases} \tag{6.7a,b}$$

Again, writing $\mathbf{T}^{(1)}$ in terms of $\mathbf{T}^{(2)}$ in a basis free form, we have

$$\mathbf{T}^{(1)} = A_1\mathbf{T}^{(2)} + A_2(\mathbf{U}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{U}) + A_3\mathbf{U}\mathbf{T}^{(2)}\mathbf{U} + A_4(\mathbf{U}^2\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{U}^2) + A_5\mathbf{U}^2\mathbf{T}^{(2)}\mathbf{U}^2 + A_6(\mathbf{U}^2\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}\mathbf{U}^2) \tag{6.8}$$

Comparing (6.7) and (6.8) in the principal axes, and solving for the unknowns A_i, we get

$$A_1 = A_3 = A_4 = A_5 = A_6 = 0; \quad A_2 = \frac{1}{2} \tag{6.9}$$

which is exactly as (2.12), as expected. If we reciprocally write $\mathbf{T}^{(2)}$ in terms of $\mathbf{T}^{(1)}$

$$\mathbf{T}^{(2)} = B_1\mathbf{T}^{(1)} + B_2(\mathbf{U}\mathbf{T}^{(1)} + \mathbf{T}^{(2)}\mathbf{U}) + B_3\mathbf{U}\mathbf{T}^{(1)}\mathbf{U} + B_4(\mathbf{U}^2\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}^2) + B_5\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}^2 + B_6(\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(1)}\mathbf{U}^2) \tag{6.10}$$

following the same procedure, we arrive at

$$B_1 = \frac{I^2\cdot III + I\cdot II^2 - II\cdot III}{III(I\cdot II - III)} \tag{6.11a}$$

$$B_2 = \frac{-I^2\cdot II}{III(I\cdot II - III)} \tag{6.11b}$$

$$B_3 = \frac{I^3 + III}{III(I\cdot II - III)} \tag{6.11c}$$

$$B_4 = \frac{1}{III} \tag{6.11d}$$

$$B_5 = \frac{I}{III(I\cdot II - III)} \tag{6.11e}$$

$$B_6 = \frac{-I^2}{III(I\cdot II - III)} \tag{6.11f}$$

6.3. The stress tensor $\boldsymbol{T}^{(m)}$ conjugate to the strain $(1/m)\,(\boldsymbol{U}^m-\boldsymbol{I})$, and $\boldsymbol{T}^{(1)}$

Relations can be easily obtained for every stress tensor $\mathbf{T}^{(m)}$ in a similar manner. However, for general m, the relations become more complicated and it is difficult to write them in terms of the three invariants, but still possible to write them in terms of stretches in a close form. As an example, the relations will be obtained for $m=3$. Writing $\mathbf{T}^{(3)}$ in terms of $\mathbf{T}^{(1)}$ in the basis free form, we have

$$\mathbf{T}^{(3)} = A_1\mathbf{T}^{(1)} + A_2(\mathbf{U}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}) + A_3\mathbf{U}\mathbf{T}^{(1)}\mathbf{U} + A_4(\mathbf{U}^2\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}^2) + A_5\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}^2 + A_6(\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(1)}\mathbf{U}^2) \tag{6.12}$$

Comparing (5.2) for $\alpha = 3$ and $\beta = 1$ with (6.12) in the principal axis, and solve for A_i, result in

$$A_1 = \frac{-1}{C}\left(2I^4 \cdot III^2 - I^3 \cdot II^2 \cdot III + I^2 \cdot II(II^3 - 4III^2) + II^2(2III^2 - II^3)\right) \tag{6.13a}$$

$$A_2 = \frac{1}{C}\left(I^3(II^3 + III^2) - 2I^2 \cdot II^2 \cdot III - I \cdot II(II^3 + III^2) + II^3 \cdot III\right) \tag{6.13b}$$

$$A_3 = \frac{-I}{C}\left(I^4 \cdot III + I^3 \cdot II^2 - 4I^2 \cdot II \cdot III + I(2III^2 - II^3) + 2II^2 \cdot III\right) \tag{6.13c}$$

$$A_4 = \frac{-1}{C}\left(I^2(II^3 - III^2) - I \cdot II^2 \cdot III - II^4\right) \tag{6.13d}$$

$$A_5 = \frac{-1}{C}\left(I^3 \cdot III + I^2 \cdot II^2 - 2I \cdot II \cdot III - II^3\right) \tag{6.13e}$$

$$A_6 = \frac{1}{C}\left(I^4 \cdot III + I^3 \cdot II^2 - 3I^2 \cdot II \cdot III - I \cdot II^3 + I^2 \cdot III\right) \tag{6.13f}$$

where

$$C = III^2\left(I^3 \cdot III - I^2 \cdot II^2 + II^3\right) \tag{6.13g}$$

Expressions for $\mathbf{T}^{(3)}$ was obtained by Guo and Man (1992) through a different mathematical procedures.

6.4. The case of $|\alpha| < 1$

Similar relations may be obtained for the stresses with indices less than 1. As an example for $\alpha = \frac{1}{2}$, from (5.2) we have

$$\mathbf{T}_{ij}^{(1/2)} = \frac{1}{2}\left(\lambda_i^{1/2} + \lambda_j^{1/2}\right)\mathbf{T}_{ij}^{(1)} \tag{6.14}$$

Multiplying (6.14) by $N_i \otimes N_j$ and summing over i and j result in

$$\mathbf{T}^{(1/2)} = \frac{1}{2}\left(\mathbf{U}^{1/2}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}^{1/2}\right) \tag{6.15a}$$

or in general, for every α

$$\mathbf{T}^{(\alpha)} = \frac{1}{2}\left(\mathbf{U}^{\alpha}\mathbf{T}^{(2\alpha)} + \mathbf{T}^{(2\alpha)}\mathbf{U}^{\alpha}\right) \tag{6.15b}$$

Alternatively, we can write $\mathbf{T}^{(1/2)}$ in a basis free form as

$$\mathbf{T}^{(1/2)} = A_1\mathbf{T}^{(1)} + A_2\left(\mathbf{U}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}\right) + A_3\mathbf{U}\mathbf{T}^{(1)}\mathbf{U} + A_4\left(\mathbf{U}^2\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}^2\right) + A_5\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}^2 + A_6\left(\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(1)}\mathbf{U}^2\right) \tag{6.16}$$

from comparison of which with (6.14) in the principal axes of $\mathbf{U}$, we can obtain A_i as

$$A_3 = A_5 = A_6 = 0$$

and

$$A_1 = \frac{III_{1/2}I_{1/2}}{I_{1/2}II_{1/2} - III_{1/2}} \tag{6.17a}$$

$$A_2 = \frac{I_{1/2}^2 - II_{1/2}}{2\left(I_{1/2}II_{1/2} - III_{1/2}\right)} \tag{6.17b}$$

$$A_4 = \frac{-1}{2\left(I_{1/2}II_{1/2} - III_{1/2}\right)} \tag{6.17c}$$

where $I_{1/2}$, $II_{1/2}$, and $III_{1/2}$ are the invariants of $\mathbf{U}^{1/2}$.

Finally, it is noted that the tensor expansion forms like (6.2), (6.4), (6.8) etc., are not unique and the unknown coefficients for all forms of expansion may be obtained in a similar manner.

7. Conclusions

In this work, general relations are found between two different stress tensors, conjugate to a class of strain measure tensors $f(\mathbf{U})$ defined in (1.4). The approach is based on the definition of Hill's principal axis method and energy conjugacy notion. The equations are first obtained between the principal components of the stresses based on two approaches explained in Sections 3 and 4, the latter of which, based on tensor algebra is quite general. These equations are then applied to find relations between conjugate stresses of the Seth–Hill strain tensors $\mathbf{E}^{(m)}$ as a subset of $f(\mathbf{U})$. The derived equations hold not only for integer but also for real indices m including zero, which corresponds to the logarithmic strain tensor $\ln(\mathbf{U})$ and its conjugate stress $\mathbf{T}^{(0)}$. The equalities are obtained for distinct as well as coalescent principal stretches. Using the relations obtained for the principal components of conjugate stresses, basis free equations are then derived between several conjugate stresses. The basis free tensor equations between two conjugate stresses is derived through the comparison of the relations between their components in the principal axes with a possible tensorial relation between the stresses expanded in the principal axes where $\mathbf{U}$ is diagonal.

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